CHAPTER 10

UNIQUENESS THEOREMS

The problems collected in this chapter represent variations on the following theme: A function possessing a definite "supply of analyticity" and vanishing "in a sufficiently intensive manner" is identically zero. The words between the quotation marks are always given an exact sense, determined by the concrete situation. The importance of this subject (almost fundamental for complex analysis in its entirety) hardly requires any explanation. In an explicit or in a masked form, it tints the entire content of this collection since in fact every approximation problem (and they are here in majority) represents, by a dual reformulation, some uniqueness problem.

Besides the questions defined along the above-mentioned model, each section of this chapter (except 2.10) touches also upon other problems.

In Sec. 1.10, in addition to the problems devoted to the zeros of various classes of analytic functions, one discusses also the problem of the solvability of the Hausdorff moment problem and also the problem of the integral representation of functions which are analytic on the Riemann surface for the logarithm. Section 3.10 is related to the theory of differential operators. Perhaps, the same refers also to Sec. 5.10, where the "uniqueness theorem" appears properly only at the end, while at the beginning one poses the question of the extention of the localness property, intrinsic to a differential operator, to operators of a more general form. In the last section one discusses the relation between the uniqueness sets for analytic functions of the Gevrey class and the peak sets for analytic functions which satisfy the Hölder condition.

The theme of this chapter is touched upon also in some sections of other chapters (3.3, 3.4, 4.4, 6.4, 7.5, 18.5, 4.6, 1.9, 1.11, 2.11, 5.11, 2.12). On the other hand, Sec. 7.10 gravitates to the analysis—synthesis problems in the spirit of Chap. 5.

1.10. SOME OPEN QUESTIONS IN THE THEORY OF REPRESENTATIONS OF ANALYTIC FUNCTIONS*

1. We denote by Ω the set of functions ω, satisfying the following conditions:

1) $\omega(x)>0$, $\omega\in C([0,1))$;

2) $\omega(0)=1$, $\int_0^1 \omega(x)dx<+\infty$;

3) $\int_0^1 |1-\omega(x)|x^{-1}dx<+\infty$.

In the theory of the factorization of functions which are meromorphic in the circle $\mathbb{D}$, developed in [1], an important part is played by the following theorem on the solvability of the Hausdorff moment problem, established in [2]: The Hausdorff moment problem

$$\mu_n=\int_0^1 x^n d\alpha(x) \quad (n=0,1,2,\ldots),$$

where

$$\mu_0=1,\ \mu_n=\left(n\int_0^1 \omega(x)x^{n-1}dx\right)^{-1} \quad (n=1,2,\ldots),$$

and $\omega\in\Omega$, $\omega\uparrow$, has a solution in the class of functions α, nondecreasing and bounded on $[0, 1]$.

Assuming that $\omega_j\in\Omega$ $(j=1,2)$, we consider the Hausdorff moment problem of the form

$$\lambda_n=\int_0^1 x^n d\beta(x) \quad (n=0,1,2,\ldots), \tag{1}$$

where

$$\lambda_0=1,\ \lambda_n=\left(\int_0^1 \omega_1(x)x^{n-1}dx\right)\left(\int_0^1 \omega_2(x)x^{n-1}dx\right)^{-1} \quad (n=1,2,\ldots). \tag{2}$$

Conjecture. The moment problem (1)-(2) has a solution in the class of functions β, nondecreasing and bounded on $[0, 1]$, or, at least in the class of functions β with a finite variation $\underset{[0,1]}{V}(\beta)<+\infty$, provided the functions ω_j $(j = 1, 2)$ are monotone on $[0, 1]$ and the function ω_1/ω_2 does not increase on $[0, 1]$.

The confirmation of this conjecture, valid in the special case when $\omega_1(x)\equiv 1$, would lead to an important result regarding the imbedding of the classes $N\{\omega_j\}$ $(j = 1, 2)$ of functions which are meromorphic in $\mathbb{D}$, considered in [1].

2. We denote by Ω_∞ the set of functions ω, satisfying the following conditions:

a) the function ω is continuous and nonincreasing on $[0, +\infty)$; $\omega(x) > 0$;

b) the integrals

$$\Delta_k=k\int_0^{+\infty}\omega(x)x^{k-1}dx, \quad (k=1,2,\ldots)$$

are finite.

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Setting $\Delta_0 = 1$, we consider the entire functions of z:

$$W_\omega^{(\infty)}(z;\xi) = \int_{|\xi|}^{+\infty} \frac{\omega(x)}{x}dx - \sum_{k=1}^{\infty}\left(\xi^{-k}\int_0^{|\xi|}\omega(x)x^{k-1}dx - \bar{\xi}^{k}\int_{|\xi|}^{+\infty}\omega(x)x^{-k-1}dx\right)\frac{z^k}{\Delta_k} \quad (0<|\xi|<\infty)$$

and

$$A_\omega^{(\infty)}(z;\xi) = \left(1-\frac{z}{\xi}\right)e^{-W_\omega^{\infty}(z;\xi)}$$

Finally, assume that $\{z_k\}_1^\infty$ $(0<|z_k|\leqslant|z_{k+1}|<\infty)$ is an arbitrary sequence of complex numbers for which

$$\sum_{k=1}^{\infty}\int_{|z_k|}^{\infty}\frac{\omega(x)}{x}dx<+\infty. \tag{3}$$

Conjecture. Under condition (3), the infinite product

$$\pi_\omega(z;z_k) = \prod_{k=1}^{\infty} A_\omega^{(\infty)}(z;z_k)$$

converges on any compactum of the complex plane which does not contain the points $\{z_k\}_1^\infty$, provided the function ω satisfies the additional condition

$$\frac{d\log\omega(x)}{d\log x}\downarrow -\infty \quad \text{for} \quad x\uparrow+\infty.$$

The validity of this conjecture in some special cases, in particular for

$$\omega(x) = \frac{\rho\sigma^\mu}{\Gamma(\mu)}\int_x^{+\infty} e^{-\sigma t^\rho} t^{\mu\rho-1}dt,$$

where $\rho(0<\rho<+\infty)$, $\mu(0<\mu<+\infty)$, and $\sigma(0<\sigma<+\infty)$ are arbitrary parameters, has been established in [3].

3. Let μ be a complex-valued function on $[0, +\infty)$, for which

$$V_\mu(r) \overset{\mathrm{def}}{=} \int_0^\infty r^t|d\mu(t)|<\infty;\quad \forall r,\ r\geqslant 0. \tag{4}$$

Then, obviously, the function f_μ,

$$f_\mu(z) = \int_0^\infty z^t\,d\mu(t),$$

is regular on the entire Riemann surface

$$G_\infty = \{z: |\operatorname{Arg} z|<+\infty,\ 0<|z|<+\infty\}$$

and we have

$$\sup_{|\varphi|<+\infty}|f_\mu(re^{i\varphi})|\leqslant V_\mu(r),\ \forall r\in[0,+\infty).$$

By virtue of this remark, it is natural to formulate the following conjecture.

Conjecture. Assume that the function f is regular on G_∞ and satisfies the condition

$$\tilde{M}_f(r) \overset{\mathrm{def}}{=} \sup_{|\varphi|<+\infty}|f(re^{i\varphi})|<+\infty,\ \forall r\in[0,+\infty).$$

Then there exists a function μ_f on $[0, +\infty)$, satisfying condition (4) and such that one has the representation

$$f(z) = \int_0^{+\infty} z^t \, d\mu_f(t), \quad z \in G_\infty. \tag{5}$$

We note that in the special case when

$$f(re^{i(\varphi+2\pi)}) = f(re^{i\varphi}), \quad \forall r \in [0, +\infty),$$

the function f admits, obviously, the series expansion

$$f(z) = \sum_{k\geq 0} a_k z^k, \quad |z| < +\infty.$$

Thus, in this case representation (5) holds with the measure μ, concentrated only at the points 0, 1, 2,... of the semiaxis $[0, +\infty)$.

4. We denote by $H_p(\alpha)$ $(0 < p < +\infty, -1 < \alpha < +\infty)$ the set of functions f, analytic in the unit circle $\mathbf{D}$, for which

$$\iint_{\mathbf{D}} (1-r^2)^\alpha |f(re^{i\theta})|^p r \, dr \, d\theta < +\infty.$$

Let $f \in H_p(\alpha)$ and let $\{\alpha_j\}_{j\geq 1}$ $(0 < |\alpha_j| \leq |\alpha_{j+1}| < 1)$ be the sequence of the zeros of the function f, their multiplicities being taken into account. We denote by n(t) the number of numbers from $\{\alpha_j\}_{j\geq 1}$ which are situated in the circle $|z| \leq t$ $(0 < t < 1)$ and let

$$N_\alpha(r) = \int_0^r \frac{n(t)}{t} dt \quad (0 < r < 1).$$

As known (see [4]), if $f \in H_p(\alpha)$ and $f(z) \not\equiv 0$, then

$$\int_0^1 (1-r)^\alpha e^{pN_\alpha(r)} dr < +\infty. \tag{6}$$

In connection with this there arises naturally the following conjecture.

Conjecture. Let $\{\alpha_j\}_{j\geq 1}$ be an arbitrary sequence from $\mathbf{D}$, satisfying condition (6). Then there exists a sequence of numbers $\{\theta_j\}_{j\geq 1}$ $(0 \leq \theta_j < 2\pi)$ and a function f_*, $f_*(z) \not\equiv 0$, $f_* \in H_p(\alpha)$, such that

$$f_*(\alpha_j e^{i\theta_j}) = 0 \quad (j = 1, 2, \ldots).$$

We note that an assertion equivalent to (6) as well as a series of theorems regarding the zeros of functions of the class $H_p(\alpha)$ for $\alpha \geq 0$ have been established in [5], long after [4].

LITERATURE CITED

1. M. M. Dzhrbashyan, "The theory of the factorization of functions, meromorphic in the circle," Mat. Sb., 79, No. 4, 517-615 (1969).
2. M. M. Dzhrbashyan, "A generalized Riemann-Liouville operator and some of its applications," Izv. Akad. Nauk SSSR, 32, 1075-1111 (1968).
3. M. M. Dzhrbashyan, "On an infinite product," Izv. Akad. Nauk Arm. SSR, Mat., 13, No. 3 (1978).
4. M. M. Dzhrbashyan, "On the problem of the representability of analytic functions," Soobshch. Inst. Mat. Mekh. Akad. Nauk Arm. SSR, 2, 3-39 (1948).
5. C. Horowitz, "Zeors of functions in the Bergman spaces," Duke Math. J., 41, 693-710 (1974).